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Calculus instructors' perspectives on effective instructional approaches in the teaching of related rates problems

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Abstract

Much research has reported on students' difficulties with solving related rates problems in calculus. In an effort to generate a resource that could potentially address some of these difficulties from a teaching standpoint, a questionnaire about effective instructional approaches related to the teaching of related rates problems, among other things, was administered to 14 veteran calculus instructors. Analysis of the responses provided by the instructors revealed that all the instructors considered the use of diagrams to be helpful when solving related rates problems. Furthermore, a majority of these instructors noted that introducing a set of steps (i.e., a guideline), during classroom instruction, that students could follow when solving related rates problems is helpful for students when working with this type of problems. These instructors further identified strengths and weaknesses in the way related rates problems are typically presented in calculus textbooks. Implications for instruction are included.

Keywords: related rates problems, calculus learning, calculus teaching

INTRODUCTION

This study uses Mkhatshwa's (2020) definition of related rates problems i.e., mathematical tasks that involve at least two instantaneous rates of change (also known as derivatives) that can be related by an equation, function, or formula. Additionally, this study uses the following distinction between geometric and non-geometric related rates problems:

> A "geometric" related rates problem is one in which the equation relating the quantities is based on a geometric structure such as the Pythagorean theorem or the volume of a shape. A "nongeometric" related rates problem is one in which the underlying equation is based on a nongeometric relationship such as a physics law (Mkhatshwa, 2020, p. 141).

Related rates problems are not only central to the study of differential calculus at the undergraduate level in the United States, but also problems that many students struggle to solve (cf. Azzam et al., 2019; Code et al., 2014; Ellis et al., 2015; Engelke-Infante, 2021; Hausknecht & Kowalczyk, 2008; Jeppson, 2019; Kottath, 2021; Martin, 2000; Mirin & Zaskis, 2019; Piccolo & Code, 2013; Taylor, 2014; White & Mitchelmore, 1996). Several researchers have called for more studies to examine students' thinking about related rates problems (cf. Engelke, 2007; Speer & Kung, 2016). In response to this call, a growing body of research has characterized the nature of difficulties exhibited by students when working with related rates problems (cf. Martin, 2000; Mkhatshwa, 2020; Mkhatshwa & Jones, 2018; White & Mitchelmore, 1996). Among other findings, these studies have found that lack of facility with calculus rules of differentiation such as the product rule serves as a stumbling block for many students when solving related rates problems (cf. Mkhatshwa, 2020; Mkhatshwa & Jones, 2018). Findings by other researchers (cf. Azzam et al., 2019; White & Mitchelmore, 1996) indicate that mathematizing (Freudenthal, 1993) related rates problems is particularly challenging for calculus students, something that often limits their ability to solve this type of problems successfully.

Undoubtedly, instructors play a significant role in providing opportunities for students to learn various calculus concepts/topics, including related rates problems, through classroom instruction. Thus, in an effort to understand students' reported difficulties with

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Contribution to the literature

- A growing body of research has reported on difficulties exhibited by students when solving related rates problems in calculus. It is also well known that most of what mathematics instructors teach during course lectures is directed by course textbooks.
- However, there is a paucity of research that has examined veteran calculus instructors with the goal of identifying effective teaching approaches related to the teaching of related rates problems, let alone examine these veterans' views on the strengths and weakness pertaining to how calculus textbooks present related rates problems, a gap in knowledge the current study seeks to address.
- The ultimate goal of this paper (i.e., main contribution to the literature) is to document effective teaching strategies that calculus instructors, especially those who have limited or no experience at all teaching calculus such as graduate teaching assistants in mathematics departments, could draw on in their teaching of related rates problems, and potentially, other related topics in calculus.

solving related rates problems, and most importantly to learn about effective instructional approaches that other instructors can adopt to help students overcome these difficulties, a questionnaire was administered to calculus instructors who have experience teaching related rates problems in calculus. Despite the fact that there is a paucity of research that has specifically examined the opportunity to learn about related rates problems provided by calculus textbooks, evidence from a related line of research indicate that most of what instructors teach during classroom instruction is directed by course textbooks (cf. Alajmi, 2012; Begle, 1973; Kolovou et al., 2009; Reys et al., 2004; Törnroos, 2005; Wijaya et al., 2015). In fact, Reys et al. (2004) posited that "… the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61). Thus, in order to gain insight on instructors' views regarding the role of textbooks in student learning of related rates problems, the questionnaire included questions that elicited instructors' perceptions on the presentation of related rates problems in calculus textbooks. The research questions investigated in this study are:

1. What do calculus instructors consider to be effective instructional strategies that could help students develop a solid understanding of related rates problems and how to solve this type of problems?
2. What do calculus instructors identify as strengths and weaknesses in the presentation of related rates problems in calculus textbooks?

RELATED LITERATURE

Students' Thinking About Related Rates Problems

Even though the aim of this study is to provide calculus instructors' perspectives on the nature of difficulties exhibited by students when solving related rates problems, and most importantly what they consider to be effective instructional strategies when teaching this type of problems, it is important to review existing literature on students' thinking about related rates problems for comparison. In other words, this will help determine whether or not the teaching strategies proposed by calculus instructors could potentially address previously-reported (in the calculus education research literature) students' difficulties when solving related rates problems.

A common finding of studies that have examined students' thinking about related rates problems is that students who are able to visualize or perform physical enactments of dynamic physical situations that can be modeled using related rates problems tend to be successful when tasked with solving such problems (cf. Carlson, 1998; Carlson et al., 2002; Monk, 1992). In their examination of students' thinking about the classical calculus ladder problem, Carlson et al. (2002) reported on students who performed physical enactments of a ladder sliding down a vertical wall in an effort to describe the vertical speed of the ladder as it slid down the wall. Carlson et al. (2002) asserted that using physical enactments of the situation provided the students with powerful tools that not only helped them solve the problem they were presented with, but also enhanced their understanding of the relationship among the quantities involved in the situation, including the vertical speed and horizontal speed of the ladder as it slid down the wall.

Another common finding of research that has investigated students' reasoning about related rates problems is that mathematizing applied related rates problems i.e., converting written prompts to mathematical structures (e.g., equations) one can operate on such as calculating derivatives is problematic for students (cf. Azzam et al., 2019; Jeppson, 2019; Martin, 2000; White & Mitchelmore, 1996). A majority of the students who participated in Martin's (2000) assessment of calculus students' ability to solve geometric related rates problems performed poorly on the tasks they were given. This researcher reported that "the poorest performance was on steps linked to conceptual understanding, specifically steps involving the translation of prose to geometric and symbolic representations" (p. 74). In addition to reporting similar results, White and Mitchelmore (1996) found that students have a propensity to treat quantities as variables that are to be manipulated and not as quantities that are to be related when solving related rates problems. A related finding by Engelke (2007) is that conceiving variables as functions of time is particularly challenging for students when working on related rates problems. Other studies have reported on students' difficulties related to making sense of quantities or relationships between quantities when solving related rates problems (cf. Azzam et al., 2019; Kottath, 2021).

Much research has reported on students' lack of facility with rules of differentiation as a major stumbling block when tasked with solving related rates problems. Specifically, some of these studies have identified implicit differentiation as problematic for students (cf. Clark et al., 1997; Engelke, 2004; Mkhatshwa & Jones, 2018; Piccolo & Code, 2013). In one study that assessed students' understanding of related rates problems, Piccolo and Code (2013) reported that most of the 300 students who participated in their study struggled with applying implicit differentiation when given a function with more than one time-dependent variable. Hare and Phillippy (2004) argued that "implicit differentiation is a difficult concept for many students to understand because the level of difficulty of the concept is higher than the level of difficulty of explicit functions" (p. 7). Recent studies on students' reasoning about related rates problems have reported on students who exhibited lack of facility with the product rule and the quotient rule (cf. Mkhatshwa, 2020; Mkhatshwa & Jones, 2018). Findings by Clark et al. (1997) indicate that using the chain rule is problematic for students when solving related rates problems.

Teaching-Learning Strategies for Calculus Topics

In as much as a growing number of studies have reported on students' difficulties when tasked with solving related rates problems in calculus, only one study (Engelke-Infante, 2021) has reported on a lesson aimed at helping students overcome known difficulties related to the teaching and learning of related rates problems. Engelke-Infante proposed "… a related rates lesson that teaches how to solve such problems [i.e., related rates problems] by pushing students toward thinking like a mathematician…" (p. 749). According to Engelke-Infante (2021), an important aspect of the lesson focuses on the benefits of using two diagrams in the solution process of a related rates problem. On a related note, Engelke-Infante (2021) argued that "many [calculus] textbooks present a procedure for their solution that is unlike how experts [calculus instructors] approach the problem and elide important details of how diagrams are used" (p. 749). In another study whose primary aim was to investigate students' quantitative reasoning when working with related rates problems, Mkhatshwa (2020) reported on high-performing calculus students who identified the creation of diagrams as helpful and crucial in the process of solving related rates problems.

A number of studies have reported on the effectiveness of adopting an inquiry-based learning approach in the teaching of calculus topics such as transcendental functions and Euler's formula (cf. Ekici & Gard, 2017; Shelton, 2017). Similarly, a growing body of research has documented benefits associated with using team-based learning or the flipped classroom in the teaching of calculus topics, including partial derivatives (cf. Peters et al., 2020; Sahin et al., 2015; Wasserman et al., 2017). Much research has reported on the benefits of using different educational mathematical technologies/software (e.g., GeoGebra and Maple) to visualize ideas/concepts in the teaching and learning of various calculus topics, including derivatives, definite integrals, and indefinite integrals (cf. Chen & Wu, 2020; Oktaviyanthi & Supriani, 2015; Salleh & Zakaria, 2016; Yimer, 2022). Other studies have reported on the benefits of integrating project-based learning in the teaching of calculus topics such as polar equations (cf. Caridade et al., 2018; Wu & Li, 2017).

Importance of Experienced Instructors' Perspectives on Teaching of Related Rates Problems

Related rates problems are typically covered in an introductory differential calculus course, commonly known as calculus I in the United States. The course is taken by thousands of students each year. In fact, according to Bressoud et al. (2013), in the fall semester alone of each year, nearly 300,000 students take calculus I in the United States. The instructors who teach this course are often at different stages of their careers, including early-career instructors (pre-tenured professors, or an equivalent rank), mid-career instructors (recently tenured professors or an equivalent rank), and late-career instructors (professors who have been tenured for a number of years, or an equivalent rank). Regardless of where calculus instructors are in their teaching career, an argument can be made that they can benefit from having a resource with ideas that other colleagues who have experience teaching calculus I have used and found effective in the teaching of related rates problems, a topic many scholars have argued is particularly challenging for calculus students (cf. Engelke, 2007; Engelke-Infante, 2021; Kottath, 2021; Martin, 2000; Mirin & Zaskis, 2019). Such a resource could potentially be beneficial especially for graduate teaching assistants who sometimes teach calculus I as the instructors of record, often serve as instructional assistants during course lectures in larger universities (with student enrollment sizes of over 20,000), where this course is typically taught by experienced instructors, or in many instances serve as recitation instructors for this course, respectively. Thus, it is my hope that the proven strategies related to the teaching of related rates problems shared by experienced calculus instructors, and presented in this paper, will serve as a resource for calculus instructors, regardless of where they may be in their teaching careers.

I am not aware of any study that has solely focused on exploring calculus instructors' experiences or perceptions on the teaching of related rates problems. The only study that can be considered to have somewhat contributed to narrowing this knowledge gap was conducted by Engelke (2007), who proposed a framework for analyzing students' work when solving related rates problem. The framework, which lists five phases (drawing diagrams, determining equations that relate quantities in related rates problems, using implicit differentiation to differentiate these equations, finding answers to questions posed in related rates problems, and checking the answers for reasonability) that one can follow when solving geometric related rates problems, emerged from interviews with three mathematics professors who had experience teaching related rates problems. Given that only three professors participated in this study, in addition to the fact that these professors were from the same institution, their opinions/perceptions related to solving related rates problems, while valuable, may not generalize to other settings. Thus, to address this limitation in Engelke's (2007) study, the present study provides perspectives of 14 experienced calculus professors from several reputable universities in the United States. It should be noted that the primary focus of Engelke's (2007) study was on investigating students' understanding of related rates problems and not on the professors' perspectives on the teaching of related rates problems.

METHODS

Questionnaire Design and Validation

This qualitative study used an eight-item online Qualtrics questionnaire (reproduced in **Appendix A**) to elicit experts' (calculus instructors) views on the teaching of related rates problems in calculus and students' difficulties related to solving this type of problems, among other things. The overarching aim of the study was to explore, through the lens of experts, what could be considered to be effective instructional approaches in the teaching of related rates problems in calculus I. With the exception of the first two items in the questionnaire (i.e., question 1 and question 2) that we included to elicit experts' background information, the rest of the questions were informed by findings from the literature reported in this study. The questionnaire was administered in the summer and fall of 2022, respectively.

My goal for designing the questionnaire is four-fold. First, to gain insight on what experts consider to be easy/straightforward or particularly challenging for students when working with related rates problems (Item 3 and Item 4 in the questionnaire). Second, to gain insight on what experts consider to be effective instructional approaches when working with related rates problems (Item 5 in the questionnaire). Third, to gain insight on experts' perceptions regarding the role of diagrams when solving related rates problems (Item 6 in the questionnaire). Fourth, to gain insight on experts' views regarding strengths and weaknesses in how calculus textbooks present related rates problems (Item 7 and Item 8 in the questionnaire).

Consistent with Martinez's (2017) definition of face and content validity, the questionnaire was assessed by two experts (i.e., undergraduate mathematics educators) for both face and content validity. Specifically, the experts were asked to evaluate the suitability of the questionnaire for the aim/purpose of the study (i.e., assess face validity). Additionally, the experts were asked to determine whether or not the questionnaire evaluates all aspects related to the teaching of related rates problems in calculus (i.e., assess content validity). The two experts judged, in their independent evaluation of the questionnaire, it to be valid with respect to the two aspects of validity (face and content) they were asked to evaluate it on.

Sampling Method and Participants

Convenience sampling was used, in two phases, to recruit the 14 experts who participated in this study. My reasons for using convenience sampling are consistent with those proposed by Stratton (2021) who posited that "convenience sampling is popular because it is not costly, not as time consuming as other sampling strategies, and simplistic" (p. 373). Stratton (2021) added, "when used to generate a potential hypothesis or study objective, convenience sampling is useful" (p. 373). Additionally, I used convenience sampling because of the lack of a sampling frame as discussed in this section. For reporting purposes, the experts have been assigned unique identifiers E1 through E14 where, for instance, E1 denotes the first expert. In the first phase, I sent out invitation emails to colleagues I know to have experience teaching calculus I in the United States. Two of the experts are colleagues at the institution I am currently affiliated with, other experts are from institutions I was previously affiliated with, other experts are colleagues I met at academic conferences on research in mathematics education, and yet other experts are colleagues in the profession that I do not know at a person level, but I am familiar with work they have done related to the teaching and learning of calculus through their research publications. In the second phase, I searched the internet using key words such as "calculus coordinator" or "calculus director." I then sent out an invitation emails to participate in the study to anyone whose contact information (email) was listed as a calculus coordinator/director in a mathematics department in the United States. In essence, while there is no sampling frame i.e., a known list of all calculus instructors and their contact information such as their email addresses in the United States, the population for this study theoretically consists of all calculus instructors in the United States.

In total, I sent out a total of 44 invitations of which 14 experts agreed to participate in the study. Of the 14 participants, only one participant responded to most of the items in the questionnaire with a 91% completion rate-the rest of the participants responded to all the questions included in the questionnaire. Two participants reported having taught at least one but no more than five sections of calculus I, six participants reported having taught at least six but no more than ten sections of calculus I, one participant reported having taught at least 11 but no more than 15 sections of calculus I, and five participants reported having taught more than 20 sections of calculus I.

At the time of the study, ten participants were affiliated with R1 institutions, three participants were affiliated with R2 institutions, and one participant was affiliated with a liberal arts college that mostly offers undergraduate degree programs, a few master's degree programs, and only one doctoral program. According to the Indiana University Center for Postsecondary Research (n. d.), R1 institutions are doctoral universities with "very high research activity", while R2 institutions are doctoral universities with "high research activity."

Data Analysis

Data for this study consists of experts' responses to Item 3 through Item 8 in the questionnaire. The data were analyzed qualitatively using thematic analysis. According to Braun and Clarke (2006), "thematic analysis is a method for identifying, analyzing, and reporting patterns (themes) within data" (p. 79). In this study, I define a theme as a similar response to the questionnaire items given by at least three experts. For example, if one expert noted implicit differentiation, another expert remarked on the chain rule, and another expert mentioned the product rule in response to Item 4 in the questionnaire that elicited experts' perceptions regarding students' difficulties when working with related rates problems, I identified lack of facility with rules of differentiation as a theme.

RESULTS

Students' Strengths When Working With Related Rates Problems

Calculating derivatives

A common theme that emerged from six experts' responses to Item 3 in the questionnaire is that calculating derivatives is often straightforward for students when working with related rates problems. Exemplary remarks made by two of the six experts in support of the aforementioned claim are: "the easy part is taking the derivative(s) of the equations" (E7) and "I think that once the students have set up the problem, they do not have trouble taking the derivative implicitly" (E11). While E7 did not specify the type of differentiation students find easy to perform when working with related rates problems, E11 specifically noted implicit differentiation, which by the way is the most common type of differentiation that is applicable when solving related rates problems. As noted in the research reviewed in this study, implicit differentiation is typically used in connection with several other rules of differentiation when solving related rates problems, including the product rule, the quotient rule, and the chain rule (cf. Clark et al., 1997; Mkhatshwa, 2020; Mkhatshwa & Jones, 2018).

Solving problems that require simple or no mathematization

Another common theme that emerged from five experts' responses to Item 3 in the questionnaire is that solving related rates problems where simple or no mathematization at all is required, or where all of the necessary information is provided is often easy for most students. Following is a reproduction of exemplary responses to Item 3 in the questionnaire given by two of the five experts:

> When they are given the equation that relates the variables (as opposed to creating one from a story). And when all of the necessary information is provided in the problem (no side calculations needed to find missing values, for example) (E2).

> The problems, where all values are provided, and they do not have to solve for anything are the easiest. Example: Find the rate of change of the area of a circle when they are provided with values for r [the radius] and dr/dt [the rate of change of the radius with respect to a time variable t]. Additionally, the problems, where the derivatives do not involve product rule or chain rule are usually easier for them (E14).

E2 noted that solving related rates problems that do not require mathematization is easy for students through his claim "… when they are given the equation that relates the variables …" A careful analysis of the expert's claim, "… as opposed to creating one [equation that relates the variables in the problem] from a story ...", suggests that solving related rates problems when mathematization of the problems is required is something challenging for students. In the case of the example given by E14 in her response to Item 3 in the questionnaire, the equation relating the quantities (i.e., the area and radius of the circle) is not given. However, determining this equation is rather simple as it only requires recalling the formula for the area (A) of a circle with radius r i.e., $A = \pi r^2$. Furthermore, E14's claim "… where the derivatives do not involve product rule or chain rule are usually easier for them" suggests that calculating derivatives involving the product or quotient rule of differentiation, respectively, is problematic for students when working with related rates problems. Both experts (E2 and E14) remarked on the ease of solving related rates problems by students when all of the necessary information needed to solve the problems are provided in the statements of the problems.

In response to Item 3 in the questionnaire, two other experts provided responses, which not only drew attention to things that come easy for students when working with related rates problems, but also noted that solving this type of problems is generally something that is quite challenging for students. In response to Item 3 in the questionnaire, one of these experts remarked: "Frankly, nothing seems straightforward with the related rates problems ... Some students can easily write equations connecting the elements of the problem, while others are clearer about identifying rates" (E8). I interpreted E8's claim that "…some students can easily write equations connecting the elements of the problem …" to mean that while mathematizing related rates problems may be particularly challenging for many students as previously reported in the research literature (cf. Martin, 20000; White & Mitchelmore, 1996), this may not necessarily be the case for all students.

Students' Difficulties When Working With Related Rates Problems

A common theme that emerged from 13 experts' responses to Item 4 in the questionnaire is that mathematizing related rates problems is particularly challenging for students. Following is a reproduction of an exemplary response to this item given by one of the 13 experts:

> While I [in response to Item 3 in the questionnaire] said it is easy to identify the variables in a scenario, when it comes to relating these variables to make equations, this is often difficult. I often encourage my students to draw diagrams, where possible to help them visualize the relationships but this is often difficult. Second, for students who are able to set the equations up correctly, a small percentage still find it hard to apply the implicit differentiation techniques properly. Furthermore, if you ask students to keep track of units all the way to the final answer, a lot of them struggle (E12).

In the preceding response, E12 not only remarked on students' difficulties with mathematizing related rates problems, but also that even when they are successful in doing this, some students struggle with applying implicit differentiation correctly, a remark made by two other experts (E3 and E5). In fact, these two experts specifically noted the chain rule, quotient rule, and product rule when commenting on students' lack of facility with implicit differentiation in the context of working with related rates problems. While not a theme, one expert listed a number of noteworthy difficulties, besides mathematization, exhibited by students when working with related rates problems as can be seen in the following reproduction of the expert's response to Item 4 in the questionnaire:

> Several things: (1) Problems involving similar triangles (because they get used to triangle problems using Pythagorean theorem and do not like the look of two fractions equal to each other). (2) Problems, where they have to solve for missing values using information in the problem. (3) Problems, where they have to decide if a certain rate of change is positive or negative. (4) Problems, where the answer does not line up with their intuition (ladder sliding down the wall–hard for them to believe the rate of change of height is different than the rate of change of the base) five. Any problem involving trig [Trigonometry]! (E14).

Among other things, a close examination of the challenges noted by E14, especially the first three challenges, suggests that solving related rates problems, where some of the information necessary to solve the problems is not obvious (i.e., it has to be deduced from the problem statement) is particularly challenging for students.

Effective Ways to Support Students Develop a Solid Understanding of Related Rates Problems

Introducing steps/guidelines

A common theme that emerged from five experts' responses to Item 5 in the questionnaire is that introducing, during course lectures, a set of steps or guideline students could follow when solving related rates problems is an effective instructional approach for helping students succeed when working with this type of problems. Following is an exemplary response to Item 5 in the questionnaire given by one of the five experts:

> I emphasize the following steps of the process (1) Make a list of the quantities for which the rate of change is either given or asked about. Introduce notation for them. (2) Find a relation between the quantities (i.e., an equation that relates the quantities). This may require first relating them to some other variable, and then eliminating it. Drawing a diagram often helps (to assist in step 2,

> which is the most difficult one, I give students a list of geometric formulas for areas and volumes). (3) Apply d/dt to both sides of the relation. This will produce a relation involving several derivatives with respect to t [a time variable]. (4) Plug in numeric values. It may be necessary to use the relation from step 2 again to find these numbers. (5) Solve for the rate of change (the derivative) that was the object [unknown quantity or answer to the question] of the question (E10).

In response to Item 5 in the questionnaire, E10 laid down a five-step guideline he has found to be helpful in his teaching of related rates problems in calculus. Among other takeaways from E10's teaching method of related rates problems is that it has a strong emphasis on quantitative reasoning i.e., making sense of quantities. In particular, this expert carefully used the language of "quantities" instead of "variables", especially in the first two steps of his five-step guideline for solving related rates problems. I interpreted the expert's usage of the language of "quantities" as opposed to the language of "variables" to mean that he values quantitative reasoning as an effective pedagogical approach that can be used to support students' learning about related rates problems.

Drawing diagrams/pictures

Another common theme that emerged from six experts' responses to Item 5 in the questionnaire is that using visuals, when possible, in the form of diagrams or pictures to portray the situation described in a related rates problem is yet another effective instructional approach that could be used to support students' thinking when solving related rates problems. Following is an exemplary response to Item 5 in the questionnaire given by one of the six experts:

> To help students visualize relationships among variables in a problem I often encourage students to draw a diagram if one is not already provided. I do this because I know that the rest of the problem would not make sense if students do not formulate equations properly. These diagrams are also helpful in understanding the problem in general. In some instances, graphing the equations in a software such as Desmos [an online advanced graphing calculator] helps them make better sense of certain aspects of the problem. I almost always ask my students to reflect on their solutions once they are done. Once they have an answer, just reread the problem to check if the answer makes sense. Finally, I often try as much as I can to make the topic relevant in real-life contexts that students are familiar with or that they like. I do this hoping that it will help them at least appreciate that the topic is useful (E12).

In stating the significance of using diagrams in her teaching of related rates problems, E12 noted three potential benefits of using diagrams in the teaching of related rates problems, namely helping students "… visualize relationships among variables [quantities]" in the problem, helping students correctly formulate equations relating quantities in such problems, and helping students understand related rates problems as indicated by her claim "… these diagrams are also helpful in understanding the problem in general." Her claim that "… in some instances, graphing the equations in a software such as Desmos [an online advanced graphing calculator] helps them make better sense of certain aspects of the problem ..." suggests that calculus instructors could find utilizing different forms of educational technologies such as Desmos helpful in supporting students' learning about related rates problems.

Because experts' responses to Item 5 in the questionnaire are at the heart of this paper, I have reproduced all the responses given by the 14 experts in **Appendix B**. It is my hope that **Appendix B** will serve as resource for calculus instructors, especially those who may have limited experience teaching related rates problems (e.g., teaching assistants in mathematics departments), that will equip them with various instructional approaches they could use as a starting point in their teaching of related rates problems in calculus.

The Role of Diagrams When Working With Related Rates Problems

A common theme that emerged from responses to Item 6 given by all the experts who participated in this study is that drawing of diagrams is important and must be encouraged among students when working with related rates problems. Following is a reproduction of exemplary responses to Item 6 in the questionnaire given by three of the 14 experts:

> Drawing diagrams is a must. I ask students to always complete the same steps, and the first step is to draw and label [diagrams]. Even for experienced instructors, it would be hard to solve many related rates problems without first drawing it. I cannot imagine students being successful if they skip this step (E1).

> Definitely encourage this. Most related rates problems have a corresponding diagram that is helpful for solving the problem. It helps with seeing how various quantities are related. It is important not just to draw the diagram but to label it correctly (E3).

> Drawing diagrams is very helpful. It helps us to see how the variables are related and what

> relationships might be useful. I know some people draw a dynamic diagram and a static diagram, but I usually just draw one. I emphasize that we should use variables for lengths that are changing and constants for ones that are not. We also put arrows to show whether the distance is increasing or decreasing as time increases (E14).

While there are a lot of similarities in the three responses, especially with regard to the overall importance of drawing diagrams when working with related rates problems, one cannot help but also notice unique elements in the experts' responses. In the case of E1's response, the expert noted that drawing diagrams is not only helpful for students but also for instructors as well as evidenced by his remark: "Even for experienced instructors, it would be hard to solve many related rates problems without first drawing them." In the case of E3's response, the expert noted the importance of not just drawing diagrams, but also labeling them correctly. In the case of E14's response, the expert noted that diagrams could be helpful in determining increasing/decreasing quantities when working with related rates problem.

It should be noted that even though all the experts generally supported the idea of using diagrams in the teaching of the related rates in calculus, one expert expressed a love-hate relationship for using diagrams in the context of working with related rates problems. Following is a reproduction of the expert's response to Item 6 in the questionnaire:

> In certain cases. Yes for triangle problems, like ladder, or pulling a boat by a rope. No for issues like inflating a spherical balloon. In the latter case, the diagram is usually gibberish and just gives students the false impression that they're making progress when they're not (E9).

In essence, I interpreted E9's response to mean that drawing diagrams could be useful when solving geometric related rates problems, but not when solving non-geometric related rates problems.

Calculus Textbooks' Weaknesses in Their Presentation of Related Rates Problems

A common theme that emerged from five experts' responses to Item 7 in the questionnaire is that textbooks provide related rates problems that are generally not realistic. Following is a reproduction of an exemplary response to Item 7 in the questionnaire given by one of the five experts:

> … The main issue is that even though there are only a few good problem setups some publishers seem to worry that they're going to infringe another's copyright. Hence, some textbooks will set up the dragging-a-rowboat problem as a truck pulling a kite. Sorry, but even though the geometry is the same the boat just makes more sense, and everyone should be able to use it (E9).

In addition to related rates problems presented in calculus textbooks not being realistic, the expert's response noted that there is a limited range of this type of problems in calculus textbooks. In other words, most of the problems tend to be similar in most commonly used calculus textbooks in the United States. A related response to Item 7 in the questionnaire given by two experts suggests that examples on related rates problems given in calculus textbooks tend to be very similar to practice problems on related rates problems presented in the same textbooks. Following is a reproduction of one of the experts' responses to Item 7 in the questionnaire:

> Students generally like to see a variety of worked-out examples, in hopes that the homework questions will mirror those. For related rates, while the process is general and can hopefully be understood by everyone, the set-up step of the question might trip them up. So if it is possible to include more examples in the textbook, it might help. Then in the homework exercises, more examples would be nice (E5).

Another exemplary response to Item 7 in the questionnaire was given by E3, one of the five experts noted earlier, who remarked: "Most of the problems are sort of contrived. Not great motivation for why we are solving these kinds of problems." E3 made an interesting observation regarding related rates problems that are not realistic i.e., such problems do not motivate students to learn about related rates problems, let alone why these problems should be solved. Arguably, this expert's observation does not only hold true for related rates problems, but for pretty much any type of problem in calculus such as optimization problems.

Another interesting response to Item 7 in the questionnaire noted that weaknesses, if any, in how calculus textbooks present related rates problems do not affect students directly: "Students do not read the textbook, so any textbook weaknesses do not affect the students directly" (E13). The expert added, "textbooks can affect the students indirectly by influencing the instructor and the delivery of the material" (E13). I interpreted this response to mean that the presentation of related rates problems in calculus textbooks tend to closely resemble how these problems are presented during classroom instruction. Arguably, textbook's presentation of mathematical content determines how such content will be presented during classroom instruction, which, to some extent, determines what students will learn. One expert did not respond to Item 7 in the questionnaire, while three other experts provided responses that suggested that they were unsure of the existence of weaknesses in the presentation of related rates problems in calculus textbooks. In fact, one of the three experts remarked: "I have not paid that much attention to this, actually, so I do not know" (E14).

Calculus Textbooks' Strengths in Their Presentation of Related Rates Problems

A common theme that emerged from responses to Item 8 given by 11 experts who participated in this study is that these experts appreciate examples, steps [for solving related rates problems], or diagrams/pictures provided by calculus textbooks in their presentation of related rates problems. Following is a reproduction of an exemplary response to Item 8 in the questionnaire given by one of the 11 experts:

> They use very good examples. They also emphasize the importance of drawing pictures, and this helps many students. The steps that these texts provide for solving related rates problems are useful for a lot of students because they at least are able to know the part(s) of the process that they do not yet understand (E12).

In the preceding response to Item 8 in the questionnaire, the expert (E12) not only remarked on the quality (i.e., very good) of examples on related rates problems provided by calculus textbooks, but she also remarked on the usefulness/helpfulness of pictures and steps/guidelines for solving related rates problems provided in these textbooks in an effort to support students' learning about related rates problems. In response to the same item, another expert who provided an exemplary response remarked: "Generally, I think the Stewart Early Transcendentals text is quite good. I like the explanations and diagrams that they provide in their examples" (E5). A third expert who gave an exemplary response to Item 8 in the questionnaire commented: "My current textbook, Thomas, has a step-by-step method that is very similar to the one I described earlier" (E9). I especially noticed that although this was not included in the questionnaire, E5 and E9 mentioned the authors, or the names of the calculus textbooks used in their institutions. One expert who provided a unique response (in the sense of not referring to examples, diagrams/pictures, or steps for solving related rates problems) to Item 8 in questionnaire also mentioned the author of the calculus textbook used at her institution:

> At our institution we use the Rogawski textbook. It does a decent job outlining the different flavors of related rates problems. I think students typically do not read the textbook, but it serves as a guide for what to present in class, particularly for instructors who do not have a lot of experience (E11).

I especially note that while E11's response is unique, it touches on three important issues related to opportunities to learn about related rates problems provided by calculus textbook. First, the fact that students do not read the textbook, an observation also made by E13, suggests that to some extent, students miss out on opportunities to learn about related rates problems provided by calculus textbooks. Second, E11's response to Item 8 in the questionnaire further suggests that some calculus textbooks such as the one used at her institution provide a variety of related rates problems as indicated by her remark: "It [the calculus textbook used at her institution] does a decent job outlining the different flavors of related rates problems." Third, E11's remark that "… it [calculus textbook used at her institution] serves as a guide for what to present in class, particularly for instructors who do not have a lot of experience" suggests that some calculus textbooks may be more helpful for calculus instructors who do not have much experience teaching related rates problems, or calculus in general, compared to other calculus textbooks.

DISCUSSION AND CONCLUSIONS

Contributing towards finding and documenting potential solutions to previously reported difficulties exhibited by students when working with related rates problems, the present study used an online questionnaire to elicit, among other things, calculus instructors' views on effective instructional approaches in the teaching of related rates problems in the United States. In what follows, I discuss findings of the present study in light of the literature reviewed earlier, followed by a discussion of the limitations of the study.

A common theme that emerged from the experts' responses when asked about their perceptions of what comes easy for students when working with related rates problems is that calculating derivatives is typically straightforward for most students. Findings from previous research on students' thinking about related rates problems indicate that among other rules of differentiation, applying the chain rule, product rule, or quotient rule, is generally difficult for students when solving related rates problems (cf. Clark et al., 1997; Mkhatshwa, 2020; Mkhatshwa & Jones, 2018). In fact, one of the experts in this study noted that the only type of related rates problems students find easy to solve are those, where "… derivatives do not involve product rule or chain rule …" (E14), suggesting that solving related rates problems that involve using the aforementioned rules of differentiation is generally difficult for students. Thus, I posit that the type of derivatives the experts were referring to would have to be simple rules of differentiation (e.g., rules for calculating derivatives of power functions and constant functions).

Nearly all the experts in this study mentioned mathematizing related rates problems when asked about their perceptions regarding what they would consider to be particularly difficult for students when tasked with solving this type of problems. This observation by the experts is consistent with findings from research that has investigated students' thinking about related rates problems (cf. Azzam et al., 2019; Jeppson, 2019; Martin, 20000; White & Mitchelmore, 1996). I note, however, that five experts in this study indicated that students do not struggle when tasked with solving related rates problems that require simple mathematization (e.g., recalling the formula for calculating the area of a circle) or those that do not require mathematization at all. Thus, I posit that solving related rates problems that require complex mathematizing, including non-routine related rates problems, is challenging for students.

When asked about effective instructional approaches they have used to support students' learning about related rates problems, or how to solve this type of problems, a majority of the experts in this study mentioned introducing a set of steps (i.e., a guideline) students could follow when working with related rates problems. One of the steps that all the experts found to be particularly helpful when solving related rates problems is drawing diagrams/pictures of the scenarios depicted in these problems, when possible. Drawing diagrams when solving related rates problems has been reported by students to useful when solving this type of problems (Mkhatshwa, 2020). Furthermore, findings of a recent study on the teaching of related rates problems whose goal was to help students think like experts (mathematicians) when solving related rates problems suggest that using two diagrams when solving related rates problems is beneficial (Engelke-Infante, 2021). Findings from a closely related line of research indicate that students who are able to visualize (e.g., through the use of mental diagrams) or perform physical enactments of dynamic situations portrayed in related rates problems tend to be successful when solving this type of problems (cf. Carlson, 1998; Carlson et al., 2002; Monk, 1992). In light of the reported benefits related to the integration of educational technologies such as dynamic software like Maple and GeoGebra in the teaching calculus ideas such as derivatives (cf. Chen & Wu, 2020; Oktaviyanthi & Supriani, 2015; Salleh & Zakaria, 2016; Yimer, 2022), I recommend the utilization of these or similar software in the teaching of related rates problems. The usage of such software could help students create dynamic diagrams/graphs depicting how different quantities involved in a related rates problem are changing in tandem such as in the classical sliding ladder problem presented in Carlson et al.'s (2002) study.

It should be noted that solving related rates problems has been identified by several scholars over the years as particularly challenging for students (cf. Azzam et al., 2019; Code et al., 2014; Ellis et al., 2015; Engelke, 2007; Engelke-Infante, 2021; Hausknecht & Kowalczyk, 2008; Jeppson, 2019; Kottath, 2021; Martin, 2000; Mirin & Zaskis, 2019; Mkhatshwa, 2020; Mkhatshwa & Jones, 2018 ; Taylor, 2014; White & Mitchelmore, 1996). Thus, in an effort to generate a resource with helpful strategies calculus instructors could draw on in their teaching of related rates problems, I have reproduced (in **Appendix B**) all the instructional strategies reported to be effective by the 14 experts who participated in this study. It is my hope that this resource (**Appendix B**) would be helpful for calculus instructors, especially those with limited calculus teaching experience such as graduate teaching assistants in mathematics departments. Given the reported benefits associated with using inquiry-based learning or the flipped classroom in the teaching of other calculus concepts such as the derivative (cf. Ekici & Gard, 2017; Peters et al., 2020; Sahin et al., 2015; Shelton, 2017; Wasserman et al., 2017), I hypothesize that the utilization of these methods of instruction in the teaching of related rates problems could be beneficial for students' learning about this type of problems.

The only known study (Engelke-Infante, 2021) that proposed a teaching approach that could be followed or adapted by calculus instructors in their teaching of related rates problems posited that calculus textbooks often do not do a good job in their presentation of related rates problems. Specifically, Engelke-Infante (2021) asserted that "many [calculus] textbooks present a procedure for their solution that is unlike how experts [calculus instructors] approach the problem and elide important details of how diagrams are used" (p. 749). A common theme that emerged from a majority of the experts' responses when asked about their perceptions regarding potential weaknesses in the presentation of related rates problems in calculus textbooks is that most of the problems in these textbooks lack a realistic context (i.e., they have camouflage contexts), something one expert (E3) argued could be demotivating students from learning about this type of problems. To some extent, this could explain why students struggle with solving this type of problems in calculus. To this end, I recommend that textbook authors include more related rates problems that have realistic and essential contexts (cf. Wijaya et al., 2015). The same recommendation goes to calculus instructors during classroom instruction on related rates problems. I further recommend that textbook selection committees in mathematics departments consider, among other things, prioritizing selecting calculus textbooks that provide opportunities for students to solve problems that have realistic and essential contexts as this has the potential to not only motivate students learn calculus, but mathematics in general.

A number of experts noted that related rates problems tend to be similar in most calculus textbooks (i.e., there is not much of a variety of this type of problems in calculus textbooks). I recommend that calculus textbooks authors create and include a wide range of related rates problems in their textbooks to maximize students' opportunities to learn about this type of problems in calculus textbooks. Two instructors asserted that to some extent, students miss out on opportunities to learn about related rates problems presented in calculus textbooks because they do not read these textbooks. In an effort to encourage students to read their textbooks, calculus instructors could, for instance, assign graded in-class and low stakes quizzes that are based on reading assignments from calculus textbooks. It should be noted that despite the aforementioned limitations in the presentation of related rates problems in calculus textbooks, a majority of the experts generally expressed appreciation for examples, steps [for solving related rates problems], or diagrams/pictures provided by calculus textbooks in their presentation of related rates problems.

Taken together, the present study characterized the nature of challenges/weaknesses calculus instructors can anticipate as well as the strengths they can anticipate and build on from students when working with related rates problems. Additionally, this study has documented teaching strategies that experienced calculus instructors have used successfully with students when working with related rates problems. Furthermore, this study provides calculus instructors' perspectives on the weaknesses and strengths in how calculus textbooks present related rates problems. Calculus instructors may consider supplementing any missing or limited opportunities observed in calculus textbooks, while taking full advantage of utilizing what experts have considered to be strengths in how calculus textbooks present related rates problems. Moreover, this study has implications for other stakeholders, including calculus textbook authors and textbook selection committees in mathematics departments. In particular, and in an effort to maximize students' learning opportunities in calculus textbooks, calculus textbooks authors should consider addressing weaknesses (e.g., lack of realistic and essential contexts in related rates problems) noted by experts in this study, while textbook selection committees in mathematics departments may want to adopt calculus textbooks that address most, or all of the weaknesses identified by experts in this study.

Finally, I discuss three limitations of the present study. First, I did not include (in the questionnaire) an item that asked the participants to provide the names of the calculus textbooks used at their institutions. Fortunately, the severity of this limitation was to some extent alleviated as three of the experts voluntarily provided either the name of the textbook or the author(s) of the textbooks used at their institutions. Consequently, I am certain that experts' observations/responses to items pertaining to calculus textbooks in this study (i.e., Item 7 and Item 8 in the questionnaire) are based on at least three different textbooks that are commonly used in the teaching of undergraduate calculus in the United States. Second, I did not interview any of the experts, something that would have created opportunities to prompt the experts to elaborate on some of their responses. Interviewing some of the experts could potentially have resulted in the generation of a "richer" dataset (i.e., a greater level of details in the experts' responses to the questionnaire items). Third, a larger sample size was not chosen for the current study. To a large extent this is due to the lack of a sampling frame, and the lack of funding that would potentially have allowed me to recruit more participants.

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APPENDIX A: QUESTIONNAIRE

1. According to the Carnegie Classification System, what is the rank (e.g., R1) of the institution you are affiliated with?
2. How many sections of calculus I have you taught?
   a. 1 to 5
   b. 6 to 10
   c. 11 to 15
   d. 16 to 20
   e. over 20
3. Drawing on your experience teaching calculus I, what have you noticed to be easy/straightforward for students when solving related rates problems?
4. Drawing on your experience teaching calculus I, what have you found to be particularly difficult/challenging for students when solving related rates problems?
5. What are some of the effective ways you have used to support students develop a solid understanding of related rates problems and how to solve them? Explain.
6. What can you say about the role of drawing diagrams, when possible, when solving related rates problems in calculus I? Do you encourage students to do this? Why or why not?
7. What do you consider to be a weakness(es), if any, in how calculus I textbooks you have used present related rates problems (e.g., in expository sections, examples, or exercises, respectively)? Explain.
8. What do you consider to be a strength(s), if any, in how calculus I textbooks you have used present related rates problems (e.g., in expository sections, examples, or exercises, respectively)? Explain.

APPENDIX B: EFFECTIVE STRATEGIES FOR TEACHING RELATED RATES PROBLEMS IN CALCULUS

Table B1. Experts' responses to Item 5 in the questionnaire

Expert	Response to Item 5: What are some of the effective ways you have used to support students develop a solid understanding of related rates problems and how to solve them? Explain.
E1	Helping students classify the problem before solving it (e.g., problem involving a right triangle). Emphasizing that the "steps" are the same for all related rates problems.
E2	Thinking about the difference between watching a movie and viewing a screen shot has been helpful. For example, the surface area of a cylinder changes because the height decreases while the radius increases. All quantities change with time. But then we take a screen shot & ask our question about that moment. I found that students who are comfortable with notation and understand what it tells them often are more confident in working through related rate problems.
E3	Describe a method with steps to follow. Emphasize and practice determining what is known and what is unknown.
E4	Practice more problems in class.
E5	I try to convince them that for word problems, when they see the word "rate," it is most likely the related rates/implicit derivative question, so if they know the process, they can apply it to this question. Beyond that, I have an in-class activity where we practice just the "set up" portion and then they state which piece they need to solve for.
E6	I usually have a student read a problem out loud and paused them frequently to write down all of the variables and what the independent variable is as well.
E7	Motivate students by choosing more interesting & current problems, performing real experiments in classroom, & making them project-based assignments than exam or quiz assignments, where students have to explain their solutions in more detail.
E8	I always give them general guidelines: Start with a diagram, identify elements that change, & elements that remain invariant. Clearly identify all given information in terms of elements of problem (which are rates?), find connection between elements, etc.
E9	I use a five step method, which consists of (1) name the variables and state precisely what each one means, (2) write in Leibniz notation the rate the problem is asking for (or the variable name if the problem isn't asking for a rate), (3) identify any values given in the problem, (4) write the precise relationship between the variables and differentiate, (5) use the information in 1-4 to solve the problem algebraically. The most important two parts of this method are: (a) getting students to state *precisely* what each variable means (e.g., "x is the number of meters that the foot of the ladder is from the wall" or "t is the number of centuries that have elapsed since the ladder started filling", and (b) Getting students to understand which quantities need a variable and which ones don't (I tell them that a quantity only needs a variable if it is capable of changing). A sub-issue is getting students to understand that a rate of change shouldn't get its own variable but should be expressed as a derivative of another variable.
E10	I emphasize the following steps of the process. (1) Make a list of the quantities for which the rate of change is either given or asked about. Introduce notation for them. (2) Find a relation between the quantities (i.e., an equation that relates the quantities). This may require first relating them to some other variable, and then eliminating it. Drawing a diagram often helps. (To assist in step 2, which is the most difficult one, I give students a list of geometric formulas for areas and volumes) (3) Apply d/dt to both sides of the relation. This will produce a relation involving several derivatives with respect to t [a time variable]. (4) Plug in numeric values. It may be necessary to use the relation from step 2 again to find these numbers. (5) Solve for the rate of change (the derivative) that was the object [unknown quantity or answer to the question] of the question.
E11	First, I have them draw a picture and identify which quantities are varying and which quantities are static. I have them label the varying quantities with variable names and the static quantities with numbers on the picture. I have them label the given rate and the desired rate using Leibniz's notation with the appropriate variables. Then I ask them to identify the mathematical relationship between the varying quantities. Next, they take derivative implicitly and plug in values.
E12	To help students visualize relationships among variables in a problem I often encourage students to draw a diagram if one is not already provided. I do this because I know that the rest of the problem wouldn't make sense if students don't formulate equations properly. These diagrams are also helpful in understanding the problem in general. In some instances, graphing the equations in a software such as Desmos [an online advanced graphing calculator] helps them make better sense of certain aspects of the problem. I almost always ask my students to reflect on their solutions once they are done. Once they have an answer, just reread the problem to check if the answer makes sense. Finally, I often try as much as I can to make the topic relevant in real-life contexts that students are familiar with or that they like. I do this hoping that it will help them at least appreciate that the topic is useful.
E13	Writing up the general steps that students can follow while solving related-rates problems (e.g., draw the picture, name the unknowns, set up the equation relating the given rate and the rate that you are solving for, etc.). Showing students many standard related rates problems in class. Giving students many homework problems on related rates. Giving students in-class worksheets on related rates.
E14	Pictures always! And most importantly, I have them write down the following for every problem (usually shorten these to "want" "when" and "know") What we want to find: When we want it to occur: What else we know from the problem.